

Geometric Measure Theory and its Applications

4/16/2007

A bit of a recap

Q1 : Why Currents and Varifolds?

Q2 : What are currents and varifolds

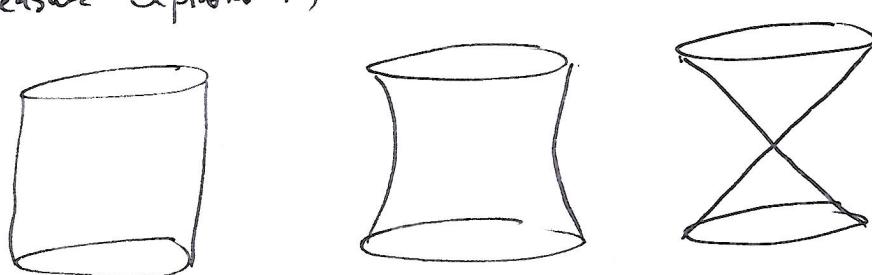
A2: Currents and varifolds are generalized surfaces

(many questions - this one included - will ~~get~~ get
answered multiple times)

A1 : When studying variational surface problems, there are several practical (mathematically practical, that is) reasons to use currents. I will give 4 examples

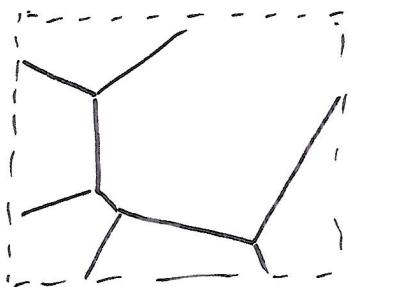
reason 1: a priori, one does not know that a minimal surface or set has any nice regularity properties. So it makes sense to pick a space of candidate minimizers general enough to include possible singularities. Therefore choosing to start with currents makes perfect sense.

reason 2: some minimal surfaces do have singularities i.e. they are not manifolds. A particular example is the 7 dim cone that appeared in a paper of Bombieri, De Giorgi and Giusti (1969). It is a minimal surface (in \mathbb{R}^8). We know this cannot happen in \mathbb{R}^n $n \leq 7$. The intuitive idea is that as $n \rightarrow \infty$ surface area gets smaller as we move towards the origin... (concentration of measure explanation)



($n-1$ dim arc of $S^{n-1} = n \alpha(n) r^{n-1} \dots$ goes to 0 rapidly)
 $\Rightarrow r$ dips below 1)

Reason 3: Partitions of open sets minimizing surface energies ... minimal surface problem and we expect to not have smooth minimizers



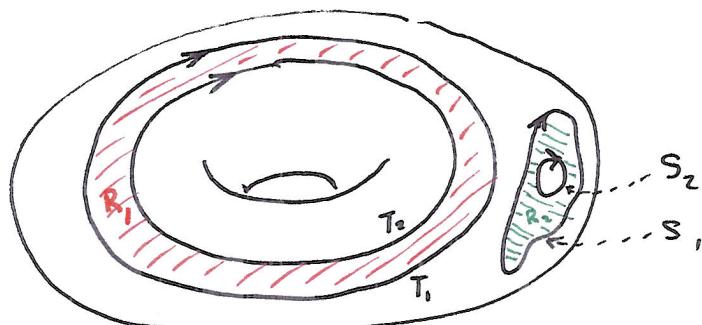
(think of crystal growth e.g.)

Integral

Reason 4: There are 7 dim^{^v} homology classes of some 14 dim compact manifolds which have no smooth representative. Currents work.

Example of "homology class"

on a torus (2-dim) the class of all 1 dim submanifolds without boundary that differ by a boundary.



$$T_1 \Delta T_2 = \partial R_1 \quad (\text{red region})$$

$$S_1 - S_2 = \partial R_2 \quad (\text{green region})$$

So T_1 & T_2 are two representatives of the same integral homology class
in the 2-Torus manifold

^{1 dim}

s_1 and s_2 are two representatives of the same homology class (but different from T_1) in the 2-torus. The example that R. Thom (? I think it was Thom) came up with was a 7 dim class in some 14 dim manifold.

A 2: First, a motivating example.

Example: generalized functions

Let f be an L^1 function on \mathbb{R}^n

Let ϕ be any smooth, compactly supported function in \mathbb{R}^n with values in \mathbb{R} .

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\phi: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\text{Then } F_f(\phi): \phi \rightarrow \int_{\mathbb{R}^n} \phi f d\mathcal{L}^n$$

is a bounded (continuous) linear functional
~~on~~ the space of "smooth, compactly supported functions" (or test functions)

~~that do not~~

But there are linear functionals that do not come from functions in this way. We can therefore think of these linear functionals as generalized functions.

Example: $\phi \rightarrow \phi(0)$

this is the famous Dirac delta "function". It corresponds to the

$f d\mathcal{L}^n$ in $\int_{\mathbb{R}^n} \phi f d\mathcal{L}^n$ getting

replaced by M_0 , the point mass at $x=0$. I.e.

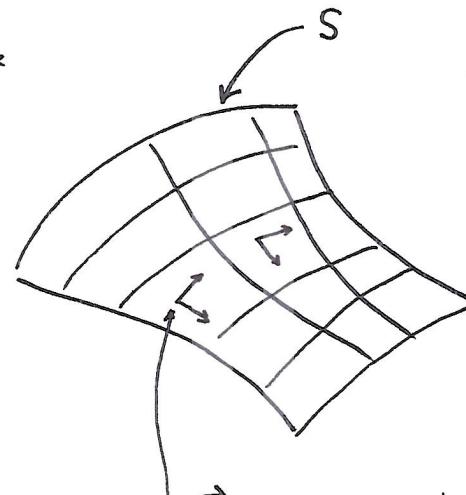
$\int f d\mathcal{L}^n \rightarrow$ measure concentrated on $x=0$ with mass = 1.

Currents:

given any smooth^{orientable} surface S , we can find the associated n -vector field \vec{S} and measure $H^n L S$

$$\|S\| =$$

such that this pair gives a member of the dual space (i.e. space of cont. linear functionals) to the space of smooth n -forms.



\vec{S} = 2-vector field on S

$$\|S\| = H^2 L S$$

Recall:

n -vector (e.g. $e_1 \wedge e_2, a e_2 \wedge e_3 + b e_4 \wedge e_5$)

n -covector (e.g. $dx_1 \wedge dx_2, a dx_1 + b dx_2 + c dx_3$)

2-covector

1-covector

n -form (e.g. $f(x_1, x_2, x_3) dx_1 \wedge dx_2$)

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$

2-form

properties: wedge product is multilinear and antisymmetric.

\Rightarrow repeated elements kill a wedge product

$$e_1 \wedge e_2 \wedge e_1 = 0$$

Define

$$T_S(\phi) = \int_S \langle \vec{s}, \phi \rangle d\|\vec{s}\|$$

$T_S(\phi)$ is a member of the space of cont. linear functionals from

$$\mathcal{D}^n(\mathbb{R}^n) = \text{space of smooth compactly supported } n\text{-forms on } \mathbb{R}^n$$

to \mathbb{R} . we denote this space

$$\mathcal{D}_n(\mathbb{R}^n) = \text{space of } n\text{-currents}$$

this ~~function~~ is in perfect analogy to the generalized functions above. $\mathcal{D}_n(\mathbb{R}^n)$ are generalized surfaces.

A2 - continued

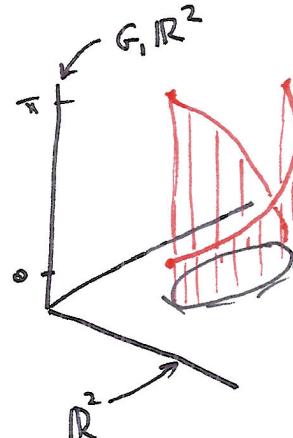
What is a varifold?

An varifold is the space of radon measures on $\mathbb{R}^n \times G_m \mathbb{R}^n$,

$$\stackrel{\text{m-dim}}{\mathbb{R}^n \times G_m \mathbb{R}^n},$$

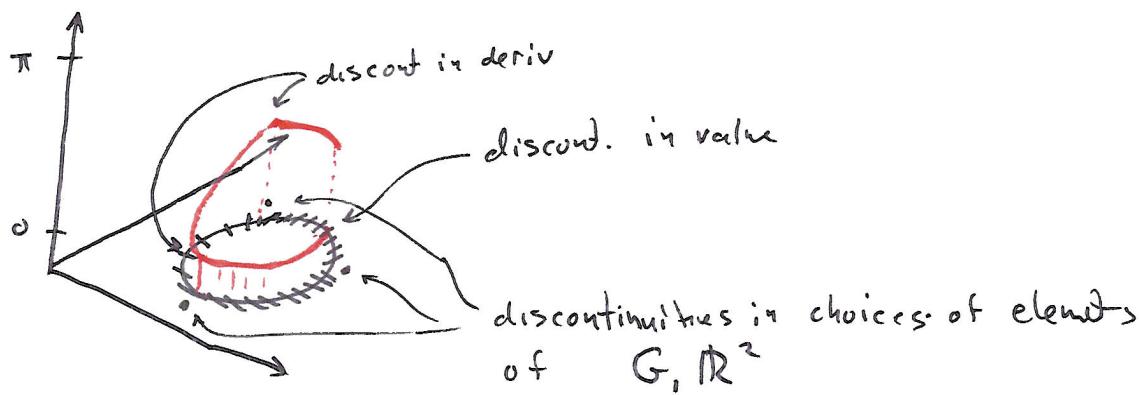
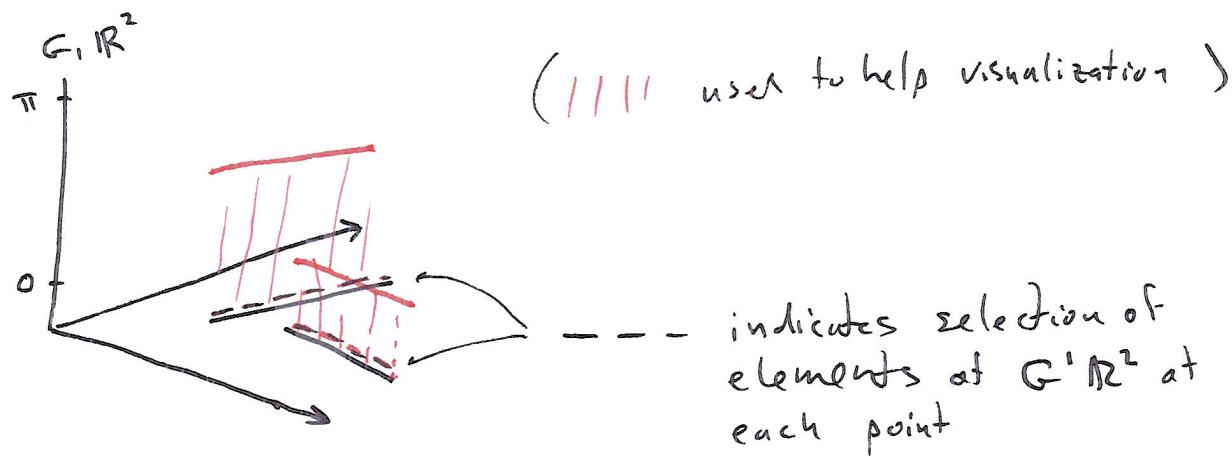
where G_m is the (compact) space of unoriented m -planes through the origin in \mathbb{R}^n .

Illustrations:



(note: what I drew in class was wrong... this is "correct")

(III) hatching used to help visualize)



In all the above **Radon** is the 1-dim measure.
radon

In the last example, the elements from G, \mathbb{R}^2 do not match the naturally suggested tangent space.

A1 why varifolds?

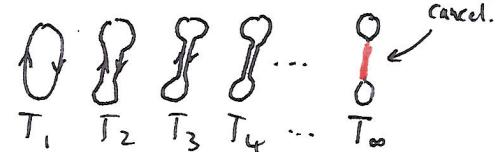
The mass (think m-volume, length in 1-currents, area in 2-currents etc)
of a current is only lower semicontinuous:

T_i a sequence of currents

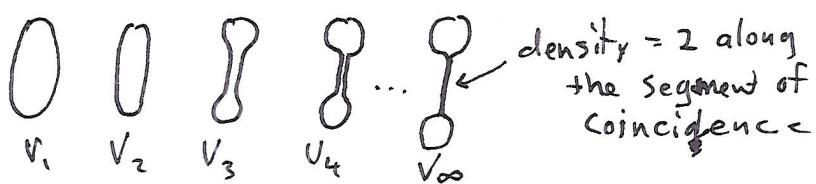
satisfies

"Mass"

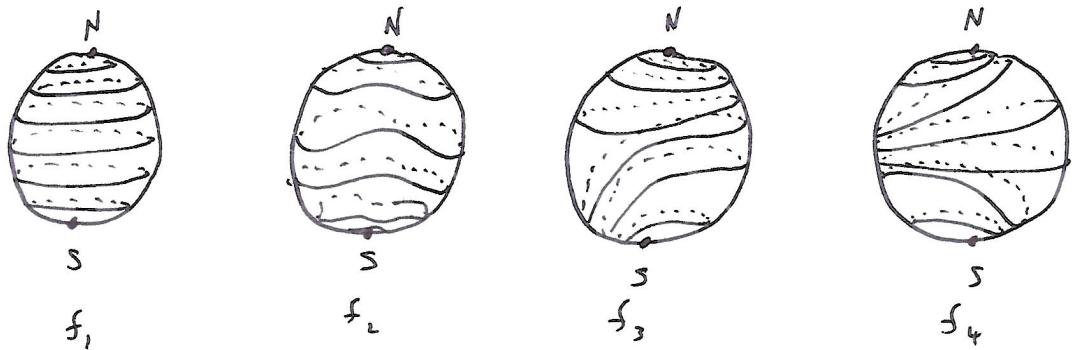
$$\liminf M(T_i) \geq M(T_\infty).$$



In cases in which we need continuity of Mass, we use varifolds



Example: Consider the maps $f: [0, 1] \rightarrow \{\text{1-submanifolds in } S^2\}$
 Examples illustrated below:



$$f_1(0) = f_2(0) = f_3(0) = f_4(0) = S$$

$$f_1(1) = f_2(1) = f_3(1) = f_4(1) = N$$

for $0 < x < 1$ $f(x)$ = curve homeomorphic to the circle and moving roughly up from S to N as x increases.

Let $M(f(x))$ be the length of $f(x)$. Call the set of all f 's homotopic to each other and $f_1 - f_4$ above, F .

$$\text{Consider } \inf_{f \in F} \left(\sup_x M(f(x)) \right)$$

convert this to
 $\hat{f} = \underset{f \in F}{\operatorname{argmin}} \left(\max_x M(f(x)) \right)$

requires continuity of $M(\cdot)$ which means \hat{f} needs to be a varifold, not a current.

\hat{f} will be a stationary ~~closed~~ curve
 i.e. a "geodesic".

Back to currents: n -vectors, n -covectors, n -forms, and boundaries

n -vectors & n -covectors: wedge products ... examples

$$\text{2-vec: } e_1 \wedge e_2$$

$$3\text{-vec: } f_{123} e_1 \wedge e_2 \wedge e_3 + f_{234} e_2 \wedge e_3 \wedge e_4$$

$$2\text{-vec: } f_{12} e_1 \wedge e_2 + f_{13} e_1 \wedge e_3 + f_{14} e_1 \wedge e_4 + f_{23} e_2 \wedge e_3 + f_{24} e_2 \wedge e_4 + f_{34} e_3 \wedge e_4$$

... replace e_i 's with e_i^* 's or dx_i 's to get
 n -covectors

$$2\text{-covec. } e_1^* \wedge e_2^* (dx_1 \wedge dx_2)$$

$$3\text{-covec. } f_{123} dx_1 \wedge dx_2 \wedge dx_3 + f_{234} dx_2 \wedge dx_3 \wedge dx_4$$

⋮

Properties:

$$e_i \wedge e_j = -e_j \wedge e_i \quad (\text{antisymmetry})$$

$$(\alpha e_1 + \beta e_2) \wedge (\gamma e_3 + \delta e_4) = \quad (\text{multilinearity example})$$

$$\alpha e_1 \wedge (\gamma e_3 + \delta e_4)$$

$$+ \beta e_2 \wedge (\gamma e_3 + \delta e_4)$$

$$= \alpha \gamma (e_1 \wedge e_3) + \alpha \delta (e_1 \wedge e_4)$$

$$+ \beta \gamma (e_2 \wedge e_3) + \beta \delta (e_2 \wedge e_4)$$

Reminder: antisymmetry
→ repeated indices
kills ~~Form~~ Form. The
suggestive parallelogram
analogy agrees.

- ✗ orthog.
- ✗ almost parallel
- ✓ repeated

orientation: any simple n -covec or n -vec is \pm any other simple n -covec or n -vec that it is a rearrangement of... i.e. e.g.

$$e_i \wedge e_k \wedge e_j = e_k \wedge e_j \wedge e_i$$

$$= -e_j \wedge e_k \wedge e_i$$

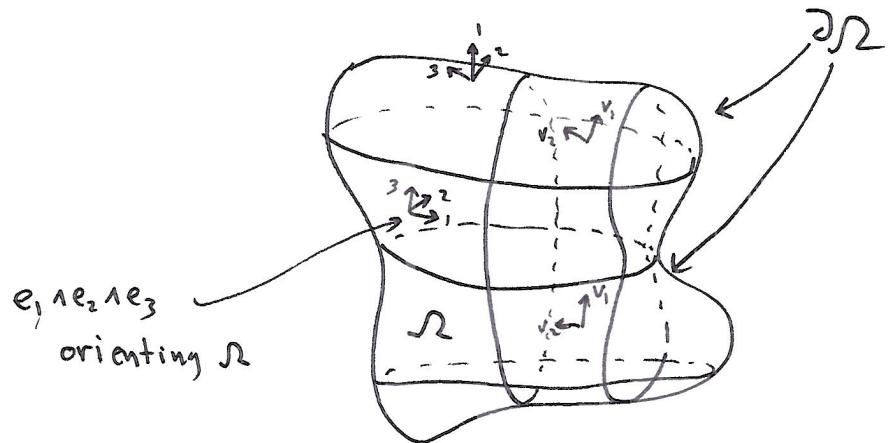
⋮

⋮

This leads to a choice when using a set of independent vectors to form an n -vec... a choice of orientation:

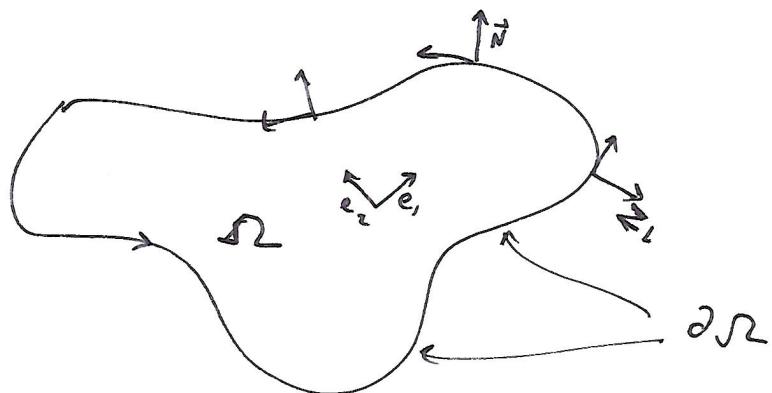
$$e_1 \wedge e_2 \wedge e_3 \text{ or } e_1 \wedge e_3 \wedge e_2 ?$$

Example 1: in \mathbb{R}^3



by lining up e_1 with the outward normal to S $e_2 \wedge e_3$ rotates into the correct orientation for ∂S .

Example 2: in \mathbb{R}^2



$e_1 \wedge e_2$ orienting S yields a counterclockwise orientation for ∂S

exterior differentiation of an n -form:

what is an n -form? simply a n -covector field!

e.g.

$$f(x, y, z) e_1^* \wedge e_2^* \text{ or } f(x, y, z) dx_1 \wedge dx_2$$

if $\omega = \sum_{\alpha} f_{\alpha} \eta_{\alpha}$ where $\eta_{\alpha} = e_{i_1}^* \wedge e_{i_2}^* \dots e_{i_n}^*$
~~i₁< i₂<...< i_n~~
then $i_1 < i_2 < \dots < i_n$

$$d\omega = \sum_{\alpha} \nabla f_{\alpha} \wedge \eta_{\alpha}$$

example of exterior differentiation of a form ω :

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

$$\begin{aligned} d\omega = & \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \frac{\partial f_1}{\partial x_3} dx_3 \right) \wedge dx_2 \wedge dx_3 \\ & + \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \frac{\partial f_2}{\partial x_3} dx_3 \right) \wedge dx_3 \wedge dx_1 \\ & + \left(\frac{\partial f_3}{\partial x_1} dx_1 + \frac{\partial f_3}{\partial x_2} dx_2 + \frac{\partial f_3}{\partial x_3} dx_3 \right) \wedge dx_1 \wedge dx_2 \end{aligned}$$

$$= \frac{\partial f_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$$

$$+ \frac{\partial f_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1$$

$$+ \frac{\partial f_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2$$

$$= (\operatorname{div} \vec{f}) dx_1 \wedge dx_2 \wedge dx_3$$

Notice: using ω and $d\omega$ from previous example,
we can rewrite the divergence theorem like
thus:

$$(\text{div}) \quad \int_{\partial\Omega} \vec{f} \cdot \vec{n} dH^2 = \int_{\Omega} \text{div } \vec{f} dH^3$$

(111)

$$(\text{stokes}) \quad \int_{\partial\Omega} \omega = \int_{\Omega} d\omega \quad \begin{array}{l} \text{when } \omega \text{ & } d\omega \text{ are} \\ \text{the form & derivation} \\ \text{in previous example} \\ \text{and } \Omega \subset \mathbb{R}^3 \end{array}$$

Stokes theorem is true for all forms and smooth enough regions.

(div \rightarrow stokes since the simple 2-vec for the tangent plane corresponding to $\vec{n} = \alpha e_1 + \beta e_2 + \gamma e_3$ is simply $\alpha e_2 \wedge e_3 + \beta e_3 \wedge e_1 + \gamma e_1 \wedge e_2$ which we ~~would~~ call \vec{n}^\perp . This can be verified by noting that $|\vec{n}| = 1$ ($\alpha^2 + \beta^2 + \gamma^2 = 1$) then $\vec{n} \wedge \vec{n}^\perp = 1 e_1 \wedge e_2 \wedge e_3$)

We now use stokes to define the boundary of a general current $T \in \mathcal{D}^n(\mathbb{R}^n)$ (or more generally $\mathcal{D}^n(U) \cup \text{open in } \mathbb{R}^n$). For surfaces or boundaries of nice subsets Ω we have the currents correspondingly to $\Omega \mapsto \partial\Omega - T_\Omega + T_{\partial\Omega}$ satisfy:

$$T_{\partial\Omega}(\omega) = T_{\Omega}^{(d\omega)}$$

For all currents then we define the boundary \hat{T} by $\hat{T}(\omega) = T(d\omega)$.

III